Dimension of a Vector Space

The dimension of a Vector Space $V(\mathbb{R})$ is the maximum possible number of linearly independant vectors that can fit in one of its linear cominations.

Base of a Vector Space

A set of linearly inpendant vectors the size of the space's dimension.

e.g. : $\mathbb{R}^3(\mathbb{R}) \longrightarrow \dim(\mathbb{R}^3) = 3$

Any vector in a Vector space can be represented as a $\overbrace{\text{L.C.}}^{\text{linear combination}}$ of any base in that space

e.g. $V^3(\mathbb{R})$ $\qquad \forall \bar{x} \in V^3 \longrightarrow \bar{x} = (x^1, x^2, x^3) = \underbrace{x^1}_{\in\mathbb{R}}\bar{e}_1 + \underbrace{x^2}_{\in\mathbb{R}}\bar{e}_2 + \underbrace{x^3}_{\in\mathbb{R}}\bar{e}_3$

Base of V^3 $\quad B = \{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$

Reference systems

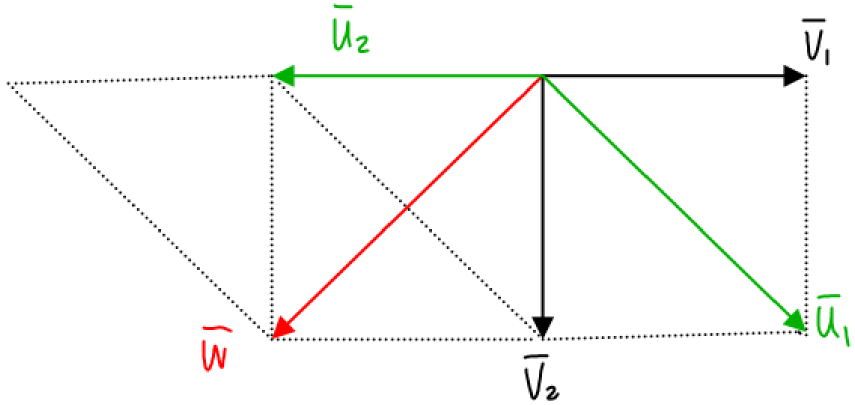
$\mathbb{R}^2(\mathbb{R})$

$$B_1 = \{\bar{u}_i\} = \{\bar{u}_1, \bar{u}_2\}$$

$$B_2 = \{\bar{v}_i\} = \{\bar{v}_1, \bar{v}_2\}$$

$$\bar{u}_1 = 1\,\bar{u}_1 + 0\,\bar{u}_2 = (1,0)_{B_1}$$

$$\bar{u}_2 = 0\,\bar{u}_1 + 1\,\bar{u}_2 = (0,1)_{B_1}$$



$$\bar{v}_1 = 1\,\bar{v}_1 + 0\,\bar{v}_2 = (1,0)_{B_2}$$

$$\bar{v}_2 = 0\,\bar{v}_1 + 1\,\bar{v}_2 = (0,1)_{B_2}$$

$$\bar{u}_1 = 1\,\bar{v}_1 + 1\,\bar{v}_2 = (1,1)_{B_2}$$

$$\bar{u}_2 = -1\,\bar{v}_1 + 0\,\bar{v}_2 = (-1,0)_{B_2}$$

$$\bar{w} = -1\,\bar{v}_1 + \bar{v}_2 = (-1,1)_{B_2}$$

$$\bar{v}_1 = 0\,\bar{u}_1 - 1\,\bar{u}_2 = (0,-1)_{B_1}$$

$$\bar{v}_2 = 1\,\bar{u}_1 + 1\,\bar{u}_2 = (1,1)_{B_1}$$

$$\bar{w} = -1\,\underbrace{(0\bar{u}_1 - 1\bar{u}_2)}_{\bar{v}_1} + 1\,\underbrace{(1\bar{u}_1 + 1\bar{u}_2)}_{\bar{v}_2} = 1\bar{u}_1 + 2\bar{u}_2 = (1,2)_{B_1}$$

$$\bar{w} = 1\,\bar{u}_1 + 2\bar{u}_2 = (1,2)_{B_1}$$

Change of Base

$$\begin{cases} \bar{u}_1 = 1\bar{v}_1 + 1\bar{v}_2 = (1,1)_{B_2} \\ \bar{u}_2 = -1\bar{v}_1 + 0\bar{v}_2 = (-1,0)_{B_2} \end{cases}$$

$$\begin{cases} \bar{v}_1 = 0\bar{u}_1 - 1\bar{u}_2 = (0,-1)_{B_1} \\ \bar{v}_2 = 1\bar{u}_1 + 1\bar{u}_2 = (1,1)_{B_1} \end{cases}$$

$$\bar{w} = -1\bar{v}_1 + \bar{v}_2 = (-1,1)_{B_2}$$

$$\bar{w} = -1\underbrace{(0\bar{u}_1 - 1\bar{u}_2)}_{\bar{v}_1} + 1\underbrace{(1\bar{u}_1 + 1\bar{u}_2)}_{\bar{v}_2} = 1\bar{u}_1 + 2\bar{u}_2 = (1,2)_{B_1}$$

$$\bar{w} = 1\bar{u}_1 + 2\bar{u}_2 = (1,2)_{B_1}$$

$$\overbrace{\begin{array}{cc} \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \\ \bar{v}_1 \quad \bar{v}_2 \end{array}}^{C} \begin{pmatrix} -1 \\ 1 \end{pmatrix}_{B_2} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}_{B_1}$$

expressed in B_1

$$\overbrace{\begin{array}{cc} \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \\ \bar{u}_1 \quad \bar{u}_2 \end{array}}^{C^{-1}} \begin{pmatrix} 1 \\ 2 \end{pmatrix}_{B_1} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}_{B_2}$$

expressed in B_2

$$\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}^{-1} = \frac{\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}^t}{1} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$$

$$\begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} = 1$$

Base change in $\mathbb{R}^3$

$\mathbb{R}^3(\mathbb{R})$ $\quad B = \{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$

$B' = \{\bar{u}_1, \bar{u}_2, \bar{u}_3\}$

$$\bar{x} \in \mathbb{R}^3 \begin{cases} \bar{x} = x^1\bar{e}_1 + x^2\bar{e}_2 + x^3\bar{e}_3 = (x^1, x^2, x^3)_B \\ \bar{x} = \alpha\bar{u}_1 + \beta\bar{u}_2 + \gamma\bar{u}_3 = (\alpha, \beta, \gamma)_{B'} \end{cases}$$

$$B \underset{C^{-1}}{\overset{C}{\rightleftarrows}} B'$$

$$\begin{cases} \bar{u}_1 = a^1\bar{e}_1 + a^2\bar{e}_2 + a^3\bar{e}_3 = (a^1, a^2, a^3)_B \\ \bar{u}_2 = b^1\bar{e}_1 + b^2\bar{e}_2 + b^3\bar{e}_3 = (b^1, b^2, b^3)_B \\ \bar{u}_3 = c^1\bar{e}_1 + c^2\bar{e}_2 + c^3\bar{e}_3 = (c^1, c^2, c^3)_B \end{cases}$$

$$C = \begin{array}{c} \begin{pmatrix} a^1 & b^1 & c^1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{pmatrix} \\ \begin{array}{ccc} \bar{u}_1 & \bar{u}_2 & \bar{u}_3 \end{array} \end{array}$$

expressed in B

The colums of C are the vetors of B' expressed in B.

How the Base Change works:

$$\boxed{\begin{gathered} (C)\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}_{B'} = \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}_B \\[2ex] (C^{-1})\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}_B = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}_{B'} \end{gathered}}$$

Inverting a matrix

$$A^{-1} = \frac{Adj(A)^T}{|A|}$$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$A^t = \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$$

$$Adj(A) = \begin{pmatrix} +A_{11} & -A_{12} & +A_{13} \\ -A_{21} & +A_{22} & -A_{23} \\ +A_{31} & -A_{32} & +A_{33} \end{pmatrix}$$

$$\boxed{A_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}}$$

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = +a_{11}\boxed{A_{11}} - a_{12}\boxed{A_{12}} + a_{13}\boxed{A_{13}}$$